1.3

(a) Let

\[ A = \{ \text{brown-eyed parents} \} \]
\[ B = \{ \text{brown-eyed children} \} \]
\[ C = \{ \text{children are heterozygotes} \} \]
\[ A_1 = \{ \text{parents are } (Xx, Xx) \} \]
\[ A_2 = \{ \text{parents are } (Xx, XX) \} \]
\[ A_3 = \{ \text{parents are } (XX, XX) \} \]

So,

\[ P(AB) = P(AB|A_1) P(A_1) + P(AB|A_2) P(A_2) + P(AB|A_3) P(A_3) \]
\[ = \left( \frac{3}{4} \right) \left[ 2p (1-p) \right]^2 + \left( \frac{1}{2} \right) \left[ 2p (1-p) (1-p)^2 \right] (2) + \left( 1 \right) \left[ (1-p)^2 \right]^2 \]
\[ = (1-p)^2 (1+2p) \]

\[ P(ABC) = P(ABC|A_1) P(A_1) + P(ABC|A_2) P(A_2) + P(ABC|A_3) P(A_3) \]
\[ = \left( \frac{1}{2} \right) \left[ 2p (1-p) \right]^2 + \left( \frac{1}{2} \right) \left[ 2p (1-p) (1-p)^2 \right] (2) + 0 \]
\[ = 2p (1-p)^2 \]

Hence,

\[ P(C|AB) = \frac{P(ABC)}{P(AB)} = \frac{2p}{2p+1} \]

(b)

\[ P(\text{Judy is Xx} | \text{n children are brown-eyes}) \]
\[ = \frac{P(\text{n children are brown-eyes} | \text{Judy is Xx}) P(\text{Judy is Xx})}{P(\text{n children are brown-eyes} | \text{Judy is Xx}) P(\text{Judy is Xx}) + P(\text{n children are brown-eyes} | \text{Judy is XX}) P(\text{Judy is XX})} \]
\[ = \frac{\left( \frac{3}{4} \right)^n \left( \frac{2p}{2p+1} \right)}{\left( \frac{3}{4} \right)^n \left( \frac{2p}{2p+1} \right) + \left( \frac{1}{2} \right)^n \left( \frac{1}{2p+1} \right)} \]
\[ = \frac{(2) (3^n) (p)}{(2) (3^n) (p) + 4^n} \]

1.4

(a)

\[ P(y > 0 | x = 8) = \frac{8}{12} \]
\[ P(y \geq 8 | x = 8) = \frac{5}{12} \]
\[ P(y \geq 8 | x = 8 \text{ and } y > 0) = \frac{5}{8} \]
(b) \( d = y - x | x \sim N(0, 14^2) \). Thus,

\[
P(y > 0 | x = 8) = P(d > -x | x = 8) = \Phi \left( \frac{8}{14} \right) = 0.72
\]

\[
P(y \geq 8 | x = 8) = P(d \geq 0 | x = 8) = 0.5
\]

\[
P(y \geq 8 | x = 8 \text{ and } y > 0) = \frac{P(y \geq 8 | x = 8)}{P(y > 0 | x = 8)} = \frac{0.5}{0.72} = 0.69
\]

1.6 Let

\[
A_1 = \{ \text{a birth with fraternal twin}\}
\]

\[
A_2 = \{ \text{a birth with identical twin}\}
\]

\[
A_3 = \{ \text{a birth with neither fraternal nor identical twin}\}
\]

\[
A = \{ \text{Elvis had a twin brother}\}
\]

Then

\[
P(A_2 | A) = \frac{P(A | A_2) P(A_2)}{P(A | A_1) P(A_1) + P(A | A_2) P(A_2) + P(A | A_3) P(A_3)}
\]

\[
= \frac{\left( \frac{1}{2} \right) \left( \frac{1}{300} \right)}{\left( \frac{1}{4} \right) \left( \frac{1}{125} \right) + \left( \frac{1}{2} \right) \left( \frac{1}{300} \right) + 0} = 0.4545
\]

1.7

\[
P(\text{win} | \text{stay}) = \frac{1}{3}
\]

\[
P(\text{win} | \text{switch}) = P(\text{the chosen box contains lesser prize}) = \frac{2}{3}
\]

So the contestant should switch the box.

2.2 \( y \): number of heads of the first 2 spins.
\( \tilde{y} \): number of additional spins until a head shows up.

We have

\[
y|\theta \sim Bin(2, \theta), \theta \sim p(\theta) = \frac{1}{2} \text{ if } \theta = 0.6 \text{ or } \theta = 0.4
\]

\[
\tilde{y}|\theta \sim Geo(\theta)
\]

So,

\[
p(\theta | y) = \frac{p(y|\theta)p(\theta)}{\sum_{\theta \in \Theta} p(y|\theta)p(\theta)} = \frac{\binom{2}{y} \theta^y (1-\theta)^{2-y} \frac{1}{2}}{\sum_{\theta \in \Theta} \binom{2}{y} \theta^y (1-\theta)^{2-y} \frac{1}{2}}
\]

\[
= \frac{\theta^y (1-\theta)^{2-y}}{0.6^y 0.4^{2-y} + 0.4^y 0.6^{2-y}}
\]
And

\[ p(\tilde{y}|y) = \sum_{\theta \in \Theta} p(\tilde{y}|\theta)p(\theta|y) \]
\[ = \sum_{\theta \in \Theta} (1 - \theta)^{\tilde{y} - 1}\theta \frac{\theta^y (1 - \theta)^{2-y}}{0.6^y 0.4^{2-y} + 0.4^y 0.6^{2-y}} \]
\[ = \frac{0.4^{\tilde{y} - y + 1} 0.6^{y+1} + 0.6^{\tilde{y} - y + 10.4^{y+1}}}{0.6^y 0.4^{2-y} + 0.4^y 0.6^{2-y}} \]

\[ \Rightarrow p(\tilde{y}|y = 0) = \frac{6}{13} (0.4^{\tilde{y}} + 0.6^{\tilde{y}}), \tilde{y} = 1, 2, 3, \ldots \]
\[ \Rightarrow E(\tilde{y}|y = 0) = \frac{6}{13} \sum_{\tilde{y}=1}^{\infty} (\tilde{y} 0.4^{\tilde{y}} + \tilde{y} 0.6^{\tilde{y}}) = \frac{6}{13} \left( \frac{0.4}{0.6^2} + \frac{0.6}{0.4^2} \right) = 2.244 \]

(by \[ \sum_{x=1}^{\infty} x(1-p)^x = \frac{1-p}{p^2} \])