STAT 9720, Winter 2008  Homework 9  Due: April 24

1. Fergusan, Chapters 21, problem 1.

2. Let \( X_1, \ldots, X_n \) be a sample from a distribution with density
\[
f(x|\theta_1, \theta_2) = \begin{cases} \frac{\theta_1}{\theta_2} e^{-x/\theta_2} & \text{for } x > 0 \\ \frac{1}{\theta_2} e^{x/\theta_2} & \text{for } x < 0 \end{cases}
\]
where \( 0 \leq \theta_1 \leq 1 \) and \( \theta_2 > 0 \).

(a) Show that \((S, K)\) is sufficient for \((\theta_1, \theta_2)\), where
\[
K = \sum_{i=1}^{N_1} I(X_i > 0), \quad S = \sum_{i=1}^{n} |X_i|.
\]
(b) Find the maximum likelihood estimate, \((\hat{\theta}_1, \hat{\theta}_2)\), of \((\theta_1, \theta_2)\).
(c) Find the Fisher information matrix, \(\mathcal{I}(\theta_1, \theta_2)\).
(d) Find the asymptotic joint distribution of \((\hat{\theta}_1, \hat{\theta}_2)\).

3. Do the following problems.

(a) Prove the following version of the Information Inequality that is valid without assuming derivatives. Assume that the support of \(f(x|\theta)\) does not depend on \(\theta\), and let \(\hat{g}(x)\) be a statistic with finite expectation, \(g(\theta) = E_{\theta}(\hat{g}(\theta))\), for all \(\theta\). Then for all \(\theta'\) and \(\theta\),
\[
\text{Var}_{\theta}(\hat{g}(X)) \geq \frac{(g(\theta') - g(\theta))^2}{\text{Var}_{\theta}(f(X|\theta')/f(X|\theta))}.
\]
(b) Dividing the numerator and denominator on the right by \((\theta' - \theta)^2\) and letting \(\theta' \to \theta\) gives the usual Information Inequality. This shows that we can define Fisher information more generally by \(\mathcal{I}(\theta) = \lim_{\theta' \to \theta}(\theta' - \theta)^{-2}\text{Var}_{\theta}(f(X|\theta')/f(X|\theta))\) Using this definition, find Fisher information for the double exponential distribution, \(f(x|\theta) = \frac{1}{2} \exp\{-|x - \theta|\}\). Does the maximum likelihood estimate achieve the Cramer-Rao bound in the limit as the sample size goes to infinity?
(c) Find Fisher information for \(f(x|\theta) = \theta I(0 < x < \theta) + (1 + \theta)I(\theta < x < 1)\), where the parameter space is \(\Theta = (0, 1)\). Can you find an estimate of \(\theta\) asymptotically as good as is indicated by Fisher information?

4. Suppose \(X_1, \ldots, X_n\) is a sample from the Cauchy \(C(\theta)\) density.

(a) Derive the Newton and Fisher scoring algorithms.
(b) The data set cauchy.dat (See this website) contains 100 observations from a \(C(\theta)\) with unknown \(\theta\). Use S-PLUS to do the following. Turn in your S-PLUS commands along with your answers.
   i. Create a plot of \(\ell_n(\theta)\).
   ii. Use both Fisher scoring and Newton’s method to estimate \(\theta\).
   iii. Compare your estimate to the sample median.