1. Let $X_1, \ldots, X_n$ be a sample from a distribution with distribution function $F(x)$, and let $M_n = \max\{X_1, \ldots, X_n\}$ be the maximum of the sample. Find a normalization, $(M_n - a_n)/b_n$, if any exists, that has a nondegenerate limiting distribution as $n \to \infty$, for the following distributions:

(a) The logistic distribution with density $f(x) = e^{-x}/(1 + e^{-x})^2$.
(b) The Pareto distribution with density $f(x) = (1/x^2)I(x > 1)$.
(c) The cosine distribution with density $f(x) = (1/2)\cos(x)I(-\pi/2 < x < \pi/2)$.

2. Suppose that $F(x) = \Phi(x - \mu)$ in Example 6 of chapter 14, so that we are sampling from a normal distribution with mean $\mu$ and variance one.

(a) Find the asymptotic distribution of $M_n$.
(b) Show that
$$\hat{\mu}_n = M_n - \sqrt{2\log n} \xrightarrow{P} \mu.$$ 
Thus $\hat{\mu}_n$ is a consistent estimate of $\mu$. What is its asymptotic efficiency relative to $\bar{X}_n$?

3. In the last problem, let $m_n = \min\{X_1, \ldots, X_n\}$.

(a) Show that
$$m_n + \sqrt{2\log n} \xrightarrow{P} \mu.$$ 
(b) Let $\tilde{\mu}_n = (m_n + M_n)/2$. Show that $\tilde{\mu}_n$ is a consistent estimate of $\mu$. What is its asymptotic efficiency relative to $\bar{X}_n$?

4. The extreme value distribution is related to logistic regression through the concept of utility maximization. Suppose you have two options to choose from, and you choose the first if $X_1 + \theta_1 > X_2 + \theta_2$, where $X_1$ and $X_2$ are independent random variables with distribution function $e^{-e^{-x}}$, and $\theta_1$ and $\theta_2$ are known constants. The idea is that $X_i + \theta_i$ is the utility for choice $i$, so the decision maker choses to maximize utility. Let $Y = 1$ for the first choice and $Y = 2$ for the second.

(a) Show that
$$P(Y = 1) = \frac{e^{\theta_1}}{e^{\theta_1} + e^{\theta_2}}.$$ 
(b) This model generalizes. Let $X_1, \ldots, X_k$ be independent with the extreme value distribution, and suppose $\theta_1, \ldots, \theta_k$ are known constants. Take $Y = i$ if and only if $X_i + \theta_i > X_j + \theta_j$ for all $j \neq i$. Show that
$$P(Y = i) = \frac{e^{\theta_i}}{\sum_{j=1}^{k} e^{\theta_j}} \text{ for } i = 1, \ldots, k.$$
5. **Extra credit** Use R (or a language of your choice) to conduct a simulation of limiting extreme value distributions. In class and homework, we found constants $a_n$ and $b_n$ such that \( (M_n - a_n)/b_n \xrightarrow{L} G(x) \) for some $G$. For a given sample size $n$, generate a large number of samples ($N$ samples) from a base distribution, and compute the sample maximum $M_n$ for each one. Then plot an estimate of the density of \( (M_n - a_n)/b_n \) (e.g., the histogram). You might take $N = 1000$ and try this for $n = 50, n = 200, \ldots$. Use the following distributions.

(a) Standard normal.
(b) Logistic.

You can experiment with different choices of $n$ and $N$. The goal is to see how large a sample is needed for the asymptotics to take effect.