Instructions:

(1) You have 50 minutes to complete the exam.

(2) Show ALL your work on the exam.

(3) You may use a 1-page summary sheet (letter size, both sides)

(4) Good Luck!
1. (25 pts) Consider the following state-space model for an AR(1) process plus noise.

\[ y_t = 1.2x_t + w_t, \quad w_t \sim iid \mathcal{N}(0, \sigma_w^2) \]
\[ x_t = 0.8x_{t-1} + v_t, \quad v_t \sim iid \mathcal{N}(0, \sigma_v^2), \]

where \( \sigma_w^2 = 0.5, \sigma_v^2 = 0.32, \) \( E(x_t) = 0, \) and \( w_t, v_t \) are independent for all \( t, t' \). The first three observations of \( y_t \) are \( y_1 = 0.3, y_2 = -2.3, \) and \( y_3 = 2.6. \)

\[ \text{a. (8 pts) Calculate } \text{var}(x_t). \]

\[ \text{b. (12 pts) Find } E(x_1|y_1). \]

\[ \text{c. (5pts) What is the “Kalman Gain” at time } t = 1? \text{ Describe heuristically the role of the Kalman Gain in the Kalman Filter.} \]
2. (25 pts) Let $x_t$ be a zero-mean stationary process with autocovariance function $\gamma_x(h)$ and power spectrum $f_x(\nu)$

   a. (15 pts) Prove that $f_x(\nu)$ can be expressed as

   $f_x(\nu) = \gamma_x(0) + 2 \sum_{h=1}^{\infty} \gamma_x(h) \cos(2\pi \nu h)$.

   b. (10 pts) Suppose that $\gamma_x(h)$ satisfies

   $|\gamma_x(h)| \leq K a^h$

   where $K$ is a positive constant and $a$ is a positive constant less than one in magnitude. Prove that

   $f_x(\nu) \leq K + 2K \left( \frac{a}{1-a} \right)$.
3. (25 pts) Let $a_t, t = 0, \pm 1, \pm 2, \ldots$, be a sequence which is absolutely summable. Let $A(\nu)$ be defined as

$$A(\nu) = \sum_{t=-\infty}^{\infty} a_t \exp(-2\pi i \nu t).$$

Show that $a_t$ can be recovered from $A(\nu)$ by taking

$$a_t = \int_{-1/2}^{1/2} A(\nu) \exp(2\pi i \nu t) d\nu.$$

(You need not prove the existence of the preceding integral.)
4. (25pts) Suppose $x_t$ is a stationary series, and we apply two filtering operations in succession, say,

$$y_t = \sum_{r=-\infty}^{\infty} a_r x_{t-r},$$

and then

$$z_t = \sum_{s=-\infty}^{\infty} b_s y_{t-s}.$$

a. (17 pts) Show the spectrum of the output is

$$f_z(\nu) = |A(\nu)|^2 |B(\nu)|^2 f_x(\nu),$$

where $A(\nu)$ and $B(\nu)$ are the Fourier transforms of the filter sequences $a_t$ and $b_t$ respectively.

b. (8 pts) What is the gain if one first filters a series with a first difference and then applies a 3-point moving average.