Chapter 5: Factorial Experiments

Up until this point, we have talked exclusively about experiments in which our primary interest surrounds a single factor. However, in many experiments we are interested in studying the joint effects of more than one factor on the outcome variables.

A traditional method for dealing with the effects of multiple factors is the one factor at a time experiment. In this case, we hold all factors except one at a fixed level. We then vary the one remaining factor to determine its effect. Next, we allow a different factor to vary while holding the remaining factors constant, and so on.

One example of this would be an experiment which considers the effect of time and temperature on a manufacturing process. We would first hold the time constant at 30 minutes and compare three temperatures, 5, 10, and 15 degrees. We would then hold the temperature constant at 5 degrees and compare two times, 20 and 30 minutes.

Notice that in this case we have some information about what happens in four cases - 5 degrees 20 minutes, 5 degrees 30 minutes, 10 degrees 30 minutes, and 15 degrees 30 minutes. We would need to have at least two samples of each type in order to be able to make any conclusions about differences between these groups. However, we can’t assess if the pattern that is observed for the three temperatures at 30 minutes is the same as the pattern that we would have observed for 20 minutes.
Factorial Arrangements

Suppose that instead of collecting two observations on each of the four combinations, we collected a single observation on each of the six possible combinations. This would give us two observations on each temperature and three observations on each time. This would permit us to estimate both the effect of temperature and time, with less observations than the previous design.

Next, suppose that we collect at least two replications of each treatment. In this situation, we can estimate not only the effects of time and temperature, but we can also assess if there is an interaction between the two factors. We will say that there exists an interaction between two factors if the difference between levels of one factor are different for different levels of another factor.

Let's draw a picture of what an interaction would look like, and also one which has no interaction.
Advantages of the Factorial Design:

- More efficient than the one factor at a time experiment.
- All of the data collected can be used in estimating all effects.
- The information gathered allows us to estimate any interactions between the factors.

Note that a \textbf{factorial experiment} is one in which every level of one factor is combined with all levels of every other factor in the experiment.

\textbf{The Model:}

We can write the model for a factorial experiment with two factors as

\[ Y_{ijk} = \mu + \tau_{ij} + \epsilon_{k(ij)} \]

where

- \( i = 1, 2, \ldots, a \) for the levels of the first factor,
- \( j = 1, 2, \ldots, b \) for the levels of the second factor,
- \( k = 1, 2, \ldots, n \) for the replicates within each \( i, j \) treatment combination,
- \( \tau_{ij} \) = the treatment effect for the \( i, j \)th treatment,
- \( \epsilon_{k(ij)} \) = the error for the \( k \)th replicate within the \( i, j \)th treatment.
Now, we mentioned that we can determine if there is an interaction between the two or more factors that we are interested in. We can thus write the treatment effects as a combination of main effects and interaction effects.

First, let us view the main effect of a factor as the effect of that factor averaged across the levels of the other factor(s). Now, we will let the interaction effect be the difference between the sum of the main effects for the factors and the effect of the treatment.

Let us denote the main effect of the first factor by $A_i$, the main effect of the second factor by $B_j$, and the interaction between the two factors as $AB_{ij}$. Using this notation, the questions that we will seek to answer are

- Is there an interaction between the two factors? This corresponds to $H_{03} : AB_{ij} = 0$ for all $i, j$.

- Is there a main effect of factor $A$? This corresponds to the hypothesis $H_{01} : A_i = 0$ for all $i$.

- Is there a main effect of factor $B$? This is equivalent to $H_{02} : B_j = 0$ for all $j$.

Recall, of course, that asking these questions is only logical in the event that we can reject a global null hypothesis, $H_0 : \tau_{ij} = 0$ for all $i, j$. Otherwise, asking these questions doesn’t make much sense (and we probably would not reject the null hypotheses either).
Interpretations:

Suppose that the global null hypothesis is rejected and we then look at the three hypotheses about main effects and interactions. Depending upon the results of these comparisons, there are several courses of action that we could take.

First, suppose that we reject the null hypothesis that there is no interaction. Then,

- We should not attempt to test for the main effects. These do not have much meaning in the presence of significant interaction because we already know that the mean for a treatment is different from the sum of two main effects.

- We could consider assessing which of the treatment means are different by looking at all possible combinations of the factors (the treatments) using the SNK procedure, for example.

Now suppose that we fail to reject the null hypothesis of no interaction. Then,

- We can test to determine if either of the main effects is significantly different from zero.

- For each main effect that is significant, we can determine which levels of that factor are different from each other using SNK or another procedure.
Once again, we will need to assess whether the assumption of independent normal errors are satisfied. We can again do this using the standard procedures that we developed earlier. In this situation, if we have several replications of each treatment, we can assess normality within each treatment group using the raw data, or we can assess it for all of the treatments together using the residuals. We should also determine if the variances across the groups are similar.

**ANOVA Rationale**

We will consider the ANOVA for two interacting factors at this point. The extension to three or more factors is straightforward, and we will talk about it briefly after completing our discussion of two factor factorials.

Recall that we wrote our model as

\[ Y_{ijk} = \mu + \tau_{ij} + \epsilon_{k(ij)}. \]

We can also write this as

\[ Y_{ijk} = \mu + A_i + B_j + AB_{ij} + \epsilon_{k(ij)}. \]

If we look at this in terms of the population means, we would have that

\[ Y_{ijk} - \mu = (\mu_i - \mu) + (\mu_j - \mu) + (\mu_{ij} - \mu_i - \mu_j + \mu) + (Y_{ijk} - \mu_{ij}). \]

Why is \( \mu_{ij} - \mu_i - \mu_j + \mu \) the correct form for the interaction term?
If we now replace all of the true means with their estimates based upon our experiment, we can write

\[ Y_{ijk} - \bar{Y}_{\ldots} = (\bar{Y}_{i\ldots} - \bar{Y}_{\ldots}) + (\bar{Y}_{.j\ldots} - \bar{Y}_{\ldots}) + (\bar{Y}_{ij.} - \bar{Y}_{i\ldots} - \bar{Y}_{.j\ldots} + \bar{Y}_{\ldots}) + (Y_{ijk} - \bar{Y}_{ij.}) \]

As with earlier designs, we are going to square both sides of this equation and sum over \(i, j, k\). All of the cross products will continue to be equal to zero, which will leave us with

\[
\sum_{i}^{a} \sum_{j}^{b} \sum_{k}^{n} (Y_{ijk} - \bar{Y}_{\ldots})^2 = \sum_{i}^{a} \sum_{j}^{b} \sum_{k}^{n} (Y_{i\ldots} - \bar{Y}_{\ldots})^2 \\
+ \sum_{i}^{a} \sum_{j}^{b} \sum_{k}^{n} (Y_{.j\ldots} - \bar{Y}_{\ldots})^2 \\
+ \sum_{i}^{a} \sum_{j}^{b} \sum_{k}^{n} (Y_{ij.} - \bar{Y}_{i\ldots} - \bar{Y}_{.j\ldots} + \bar{Y}_{\ldots})^2 \\
+ \sum_{i}^{a} \sum_{j}^{b} \sum_{k}^{n} (Y_{ijk} - \bar{Y}_{ij.})^2
\]

We will be able to break down the degrees of freedom in a similar manner with

\[(abn - 1) = (a - 1) + (b - 1) + (a - 1)(b - 1) + ab(n - 1).\]
To see why these are the appropriate degrees of freedom, first consider the main effects. For each we have the number of classes minus one. This is consistent with everything that we have seen thus far.

Next, consider the interaction term. We have a total of $ab$ treatments. This would mean that we have $ab - 1$ degrees of freedom for the treatment, $a - 1$ of which are for the first main effect, and $b - 1$ of which are for the second main effect. The remainder is $ab - a - b + 1 = (a - 1)(b - 1)$ degrees of freedom.

Finally, for the error term, when we had a completely randomized design, we said that the degrees of freedom for error were $N - k$, where $N$ was the total number of samples (now $N = abn$) and $k$ was the number of treatments. Now, $k = ab$ and the error degrees of freedom are $abn - ab = ab(n - 1)$.

Your book provides calculation formulas to determine the sums of squares by hand - we will not cover this. However, look through the example provided there. Also, let us write the general form for the ANOVA table for a factorial experiment with two factors.
Now, let’s do some examples where we will figure out the degrees of freedom for the main effects, interactions, and the errors.

Suppose that Samuel is interested in studying the effects of age, temperature, and location on the rating that individuals give a glass of wine. He is interested in studying three ages, two temperatures, and four states of origin. If he gets three individuals to rate each of the combinations of the three factors, can you fill in the degrees of freedom for the ANOVA table?

Next, suppose that Ginny is studying the durability of canvas. She has two different fiber diameters, two different weaving patterns, two different fiber sources, and five canvas samples per combination, can you find the degrees of freedom for the ANOVA table?
One Observation Per Treatment

Your book spends a little time talking about what happens in the case where we have only a single observation on each treatment. Although this is not a good way to do a factorial design, we will talk about the implications of this type of design.

First, notice that when there is only one replication per treatment, the subscript $k$ is not needed and we can write the model as

$$ Y_{ij} = \mu + A_i + B_j + AB_{ij} + \epsilon_{ij}. $$

However, we don’t have any way to distinguish between the interaction effect and the error in this case - they are confounded.

Then, the only reasonable thing to do in this case is to assume that there is no interaction between the factors. Hopefully we have some prior reason for believing this to be the case. In this case, we can write the model as

$$ Y_{ij} = \mu + A_i + B_j + \epsilon_{ij}. $$

Notice that this form looks similar to that for the randomized complete block design, and the analysis is run in the same way. However, in this case, we have two factors that we are interested in, as compared to a single factor of interest in the randomized complete block design.

Your book also describes in more detail how to analyze data from a factorial experiment with only one replicate per treatment. Please read this if you believe that it will be of use to you.
Let's look at an example that we will analyze using SAS.

First, suppose that we have have three types of gasoline and two types of motor oil that we can use to run the university's lawn mowing equipment. Suppose that we run our lawn mower three times with each of the combinations of the three types of gas and two types of motor oil. Assume that the data are summarized below in the SAS code:

```sas
options nonumber linesize=80;
data example;
input gas motor $ miles @@;
cards;
89 A 23 89 A 26 89 A 22
91 A 24 91 A 26 91 A 27
93 A 25 93 A 27 93 A 30
89 B 19 89 B 22 89 B 22
91 B 22 91 B 23 91 B 24
93 B 21 93 B 22 93 B 27 ;
proc glm;
class gas motor;
model miles=gas|motor /P;
output out=exampleout p=mileshat;
means motor/snk;
data exampleout2;
set exampleout;
resid= miles - mileshat;
proc univariate plot normal;
run;
```
Suppose that we have a very simple experiment where we are interested in comparing two types of clay and two casting procedures to determine the strength of stoneware. If we can only place four cups into our kiln in a particular run, and we wish to do five replicates of each of the four combinations of treatment, what kind of design should we use? How would we analyze it using SAS?

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What do you think this design is? Can you figure out how to analyze this using SAS?