3.2 The Acceptance-Rejection Algorithm in generating continuous r.v.'s

This is a very powerful method for generating continuous r.v.'s (probably the best one).

Suppose that we have a method for generating a r.v. having p.d.f. \( g(x) \). Then, we can accept a generated as a value from the random variable \( X \sim f(x) \), with a probability proportional to \( \frac{f(x)}{g(x)} \).

More precisely, letting \( c \) denote a constant such that
\[
\frac{f(y)}{g(y)} \leq c, \text{ for all } y,
\]
that is \( c = \max_y \left\{ \frac{f(y)}{g(y)} \right\} \), the Rejection Method consists of the following steps:

**Step 1:** Generate the value of \( Y \sim g(y) \)

**Step 2:** Generate a random number \( U \sim U(0, 1) \).

**Step 3:** If \( U < \frac{f(y)}{g(y)} \) set \( X = Y \), the generated value from \( f(x) \), otherwise return to step 1.

**Note 3.4** What is the probability we accept a certain value \( y \) of \( Y \) in a single iteration? Similarly as we showed for the discrete case one can show at each iteration, the probability that we accept the value generated (no matter which one it was) is \( \frac{1}{c} \) (common for all iterations).

**Note 3.5** How many iterations does it take to accept a value? As in the discrete case, each iteration is thought of as a Bernoulli trial, with Success when we accept and \( P(\text{Success}) = \frac{1}{c} \). Thus, \( N \) : the number of iterations until the first accepted value, is a Geometric r.v. with probability of success \( \frac{1}{c} \). Hence, the average number of iterations until we generate a valid value from \( f(x) \) is \( c \).

3.2.1 Generating a Gamma r.v.

Assume now that \( X \sim G(a, b) \), where \( a, b > 0 \), both real. The case of an integer \( a \) was studied through the inverse transform method. We have the target p.d.f. being
\[
f(x|a, b) = \frac{x^{a-1}e^{-\frac{x}{b}}}{b^a\Gamma(a)}, a, b, x > 0.
\]

A simple way to go about applying the Rejection method here, is to consider an exponential with mean equal to the mean of the Gamma, i.e., \( Y \sim Exp \left( \frac{1}{\mu} = \frac{1}{ab} \right) \), where \( \mu = E(Y) = ab \), the mean of the Gamma\((a, b)\) r.v. Then
\[
h(x) = \frac{f(x|a, b)}{g(x|b)} = \frac{x^{a-1}e^{-\frac{x}{b}}}{b^a\Gamma(a)} \frac{1}{ab} e^{-\frac{y}{b}} = \frac{ax^{a-1}e^{-\frac{x(a+1)}{ab}}}{b^{a-1}\Gamma(a)} = \frac{ax^{a-1}e^{-\frac{x(a+1)}{ab}}}{b^{a-1}\Gamma(a)}
\]
Now let's find \( c \). We have

\[
\frac{dh(x)}{dx} = \frac{a}{b^{a-1} \Gamma(a)} \left[ (a-1)x^{a-2}e^{-\frac{x}{ab}} + x^{a-1}e^{-\frac{x}{ab}} \left( -\frac{a-1}{ab} \right) \right]
\]

\[
= \frac{a(a-1)e^{-\frac{x(a-1)}{ab}}}{b^{a-1} \Gamma(a)} \left[ x^{a-2} - \frac{x^{a-1}}{ab} \right]
\]

\[
= \frac{a(a-1)e^{-\frac{x(a-1)}{ab}}x^{a-2}}{b^{a-1} \Gamma(a)} \left[ 1 - \frac{x}{ab} \right],
\]

and thus

\[
\frac{dh(x)}{dx} = 0 \Rightarrow x_0 = ab.
\]

Then

\[
c = \max_x \left\{ \frac{f(x)}{g(x)} \right\} = \frac{f(x_0)}{g(x_0)} = \frac{a(ab)^{a-1}e^{-\frac{a(b-1)}{ab}}}{b^{a-1} \Gamma(a)},
\]

or

\[
c = \frac{a^a e^{-a+1}}{\Gamma(a)}
\]

and hence the algorithm now becomes:

**Step 1:** Generate the value of \( Y \sim \text{Exp}\left(\frac{1}{ab}\right) \)

**Step 2:** Generate a random number \( U \sim U(0,1) \).

**Step 3:** Since

\[
\frac{f(y)}{cg(y)} = \frac{e^{-\frac{y}{ab}}}{\frac{y^{a-1}e^{-\frac{y(a-1)}{ab}}}{\Gamma(a)}} = \frac{y^{a-1}e^{-\frac{y(a-1)}{ab}}}{a^{a-1}b^{a-1} \Gamma(a)}
\]

then if \( U < \frac{f(y)}{cg(y)} = \frac{y^{a-1}e^{-\frac{y(a-1)}{ab}}}{a^{a-1}b^{a-1} \Gamma(a)} \), set \( X = Y \), the generated value from \( G(a,b) \), otherwise return to step 1.

3.2.2 Generating a Standard Normal r.v.

Assume now that \( X \sim N(0,1) \). We have the target p.d.f. being

\[
f(x) = (2\pi)^{-\frac{1}{2}} e^{-\frac{x^2}{2}}, \quad x \in \mathbb{R}.
\]

First we consider \(|X|\) which has p.d.f.

\[
f(x) = 2(2\pi)^{-\frac{1}{2}} e^{-\frac{x^2}{2}}, \quad x > 0,
\]

and we will use an exponential \( Y \sim \text{Exp}(1) \), as the source distribution to initially generate from \(|X|\). We have

\[
\frac{f(x)}{g(x)} = \frac{2(2\pi)^{-\frac{1}{2}} e^{-\frac{x^2}{2}}}{e^{-x}} = \sqrt{\frac{2}{\pi}} e^{\frac{x^2}{2}}
\]

and since

\[
\frac{d}{dx} \frac{f(x)}{g(x)} = \sqrt{\frac{2}{\pi}} e^{\frac{x^2}{2}} (1 - x),
\]

the maximum occurs at \( x = 1 \), i.e.,

\[
c = \max_x \left\{ \frac{f(x)}{g(x)} \right\} = \frac{f(1)}{g(1)} = \sqrt{\frac{2}{\pi}} e^{\frac{1}{2}} = \sqrt{\frac{2e}{\pi}}
\]
Thus, in order to generate from the absolute value of the standard normal $|X|$, and since

\[
\frac{f(x)}{cg(x)} = \sqrt{\frac{2}{\pi}} e^{\frac{x^2}{2}} \frac{1}{\sqrt{2e}} = e^{\frac{x^2}{2} - \frac{1}{2}} = e^{-\frac{(x-1)^2}{2}},
\]

the algorithm becomes:

**Step 1:** Generate the value of $Y \sim \text{Exp}(1)$

**Step 2:** Generate a random number $U \sim U(0, 1)$.

**Step 3:** If

\[
U < \frac{f(y)}{cg(y)} = e^{-\frac{(y-1)^2}{2}},
\]

set $X = Y$, the generated value from $|X|$, otherwise return to step 1.

Now to obtain a generated value from $X \sim N(0, 1)$, we simply take $X = Y$ or $X = -Y$, with equal probability. The next step is as follows:

**Step 4:** Generate a random number $U_1 \sim U(0, 1)$, and if $U_1 \leq .5$, set $X = -Y$, otherwise set $X = Y$.

**Note 3.6** Since $-\ln U \sim \text{Exp}(1)$, and

\[
\begin{align*}
U &< e^{-\frac{(y-1)^2}{2}} \iff \\
\ln U &< -\frac{(y-1)^2}{2} \iff \\
-\ln U &> \frac{(y-1)^2}{2},
\end{align*}
\]

we may combine steps 1,2 and 3 into the steps:

**Step 1:** Generate independent $Y_1, Y_2 \sim \text{Exp}(1)$.

**Step 2:** If

\[
Y_2 > \frac{(Y_1 - 1)^2}{2},
\]

then set $Y = Y_2 - \frac{(Y_1 - 1)^2}{2}$, and proceed to step 3, otherwise go to step 1.

**Step 3:** Generate $U \sim U(0, 1)$ and set

\[
X = \begin{cases} 
Y_1, & U \leq .5 \\
-Y_1, & U > .5 
\end{cases}
\]

This one is much faster since we dont have to exponentiate in the condition in step 2. Moreover, note that the random variables $X$ and $Y$ generated this way are independent with $X \sim N(0, 1)$ and $Y \sim \text{Exp}(1)$.

**Note 3.7** To generate from $X \sim N(\mu, \sigma^2)$, we generate $Z \sim N(0, 1)$, and take $X = \mu + \sigma Z$.

3.2.3 Generating a Beta r.v.
Assume now that $X \sim B(a,a)$. We have the target p.d.f. being

$$f(x|a,b) = \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)}, \ x \in (0,1), \ a,b > 0.$$ 

Clearly, we can use a $U(0,1)$ as the source distribution, i.e., $g(x) = 1, \ 0 \leq x \leq 1$. We have

$$\frac{f(x)}{g(x)} = \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)},$$

and since

$$\frac{d}{dx} \frac{f(x)}{g(x)} = \frac{(a-1)x^{a-2}(1-x)^{b-1} - (b-1)x^{a-1}(1-x)^{b-2}}{B(a,b)}$$

$$= \frac{x^{a-2}(1-x)^{b-2}}{B(a,b)} [(a-1)(1-x) - (b-1)x]$$

$$= \frac{x^{a-2}(1-x)^{b-2}}{B(a,b)} [a-1 + (-b+1-a)x]$$

$$= \frac{x^{a-2}(1-x)^{b-2}}{B(a,b)} [a-1 + (2-b-a)x],$$

the maximum occurs at $x = \frac{1-a}{2-b-a}$ (provided of course that a value in $(0,1)$), i.e.,

$$c = \max_x \left\{ \frac{f(x)}{g(x)} \right\} = \frac{\frac{1-a}{2-b-a}}{\frac{1-a}{2-b-a}}$$

$$= \frac{\left(\frac{1-a}{2-b-a}\right)^{a-1}}{B(a,b)} \left(1 - \frac{1-a}{2-b-a}\right)^{b-1}.$$

Since

$$\frac{f(x)}{cg(x)} = \frac{x^{a-1}(1-x)^{b-1}}{B(a,b)} \frac{1}{\left(\frac{1-a}{2-b-a}\right)^{a-1} \left(1 - \frac{1-a}{2-b-a}\right)^{b-1}}$$

$$= \frac{x^{a-1}(1-x)^{b-1}}{\left(\frac{1-a}{2-b-a}\right)^{a-1} \left(1 - \frac{1-a}{2-b-a}\right)^{b-1}},$$

the algorithm becomes:

**Step 1:** Generate the value of $U_1 \sim U(0,1)$

**Step 2:** Generate a random number $U \sim U(0,1)$.

**Step 3:** If

$$U < \frac{f(u_1)}{cg(u_1)} = \frac{u_1^{a-1}(1-u_1)^{b-1}}{\left(\frac{1-a}{2-b-a}\right)^{a-1} \left(1 - \frac{1-a}{2-b-a}\right)^{b-1}},$$

set $X = U_1$, the generated value from $X$, otherwise return to step 1.