3.5 Generating a Multivariate normal random vector

In this lecture we are interested in generating random vectors from a Multivariate Normal distribution with mean $\mu$ and variance-covariance matrix $\Sigma$. Recall the p.d.f. of the $p$-variate normal distribution is given by

$$f(x|\mu, \Sigma) = \frac{1}{(2\pi)^{p/2} |\Sigma|^{1/2}} \exp\left\{-\frac{1}{2} (x - \mu)^T \Sigma^{-1} (x - \mu) \right\},$$

where $\mu \in \mathbb{R}^p$, $\Sigma$ is positive (semi) definite, with $x \in \mathbb{R}^p$. The case of singular $\Sigma$, i.e., determinant zero yields what we call the singular $p$-variate normal and where the density above doesn’t exist.

Since we can always generate vectors of independent $N(0, 1)$ normal random variables, for example taking the vector to consist of a random sample, then we can use Choleski decomposition to obtain a realization from a general multivariate normal.

Since $\Sigma$ is positive definite, then it has a Choleski decomposition, namely there exists upper triangular matrix $C$ such that

$$\Sigma = C^T C.$$

Now assume that $z \sim N_p(\mathbf{0}, I_p)$. Then we can easily see that $x = \mu + C^T z$, is distributed according to a $N_p(\mu, C^T I_p C = \Sigma)$, the target distribution.

3.5.1 Generating a Non-Central Chi-Square

Recall that if $x \sim N_p(\mu, I_p)$, then $\chi = x^T x \sim \chi_p^2(\mu^T \mu)$, the Non-Central $\chi^2$ with $p$-degrees of freedom and non-centrality parameter $\mu^T \mu$. Moreover, the p.d.f. of $\chi$ can be written as a Poisson($\lambda = \mu^T \mu$) mixture of central $\chi^2$ r.v.’s, with degrees of freedom $p + 2k$, for $k = 0, 1, \ldots$. Note: $E(\chi^2) = p + \lambda$.

3.5.2 Generating a Wishart distribution

Now assume that that $x_1, \ldots, x_n \sim N_p(\mathbf{0}, \Sigma)$, and define the random matrix $Z = [x_1, \ldots, x_n]$ (with vec$(Z) \sim N_{np}(\mathbf{0}, I_n \otimes \Sigma)$), $n \geq p$ (otherwise singular). Then if we define the random matrix $W = ZZ^T = \sum_{i=1}^n x_i x_i^T$ we have $w \sim \mathcal{W}_p(n, \Sigma)$, the Wishart distribution of $p$-dimensions, $n$-degrees of freedom and parameter matrix $\Sigma$. Note: $E(W) = n\Sigma$. 

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