


4. (Neyman and Scott (1948) Problem). Let $Y_{ij}, i = 1, \cdots, n, j = 1, 2$ be independent observation, from $N(\mu_i, \sigma^2)$.

(a) Find $(\mu_M, \sigma^2_M)$, the maximum likelihood estimate of $(\mu, \sigma^2)$, where $\mu = (\mu_1, \cdots, \mu_n)$.

(b) Is $\sigma^2_M$ a consistent estimate of $\sigma^2$?

(c) Find an unbiased estimate of $\sigma^2$ based on $\sigma^2_M$. Is it consistent estimate of $\sigma^2$?

(d) Consider the class of prior $p_a(\mu, \sigma) = \frac{1}{\sigma^a}$, for some fixed $a \geq 0$. (This is equalent to $p(\mu, \sigma^2) \propto (\sigma^2)^{-\frac{a+1}{2}}$). Find the posterior mean of $\sigma^2$.

(e) Under the Jeffreys prior (i.e., $p_{n+1}$), write the posterior mean of $\sigma^2$? Is it consistent?

(f) Under the reference prior, (i.e., $p_1$), write the posterior mean of $\sigma^2$? Is it consistent?

5. Assume that $Y_1$ has Weibull $(\theta, \beta)$ distribution with pdf.

$$f(y|\theta, \beta) = \frac{\beta y^{\beta-1}}{\theta^\beta} \exp\{-y/\theta^{\beta}\}, \quad y > 0.$$  

Here $\beta > 0$ is the shape parameter and $\theta$ is the characteristic life.

(a) Find the Fisher information matrix for $(\theta, \beta)$.

(b) Find Jeffreys’ prior for $(\theta, \beta)$.

(c) Find the reference prior for $(\theta, \beta)$ when $\theta$ is the parameter of interest.

(d) Find the reference prior for $(\theta, \beta)$ when $\beta$ is the parameter of interest.

(e) Let $Y = (Y_1, \cdots, Y_n)$ be a random sample from Weibull $(\theta, \beta)$ and $n > 1$. Consider the noninformative prior of the form

$$p_b(\theta, \beta) \propto \frac{1}{\theta \beta^b}.$$  

for some fixed $b \geq 0$. Write the conditional posterior of $\theta$ given $\beta$ and the marginal posterior density of $\beta$.  

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