1. (6.41)

(a) Measurement equivariant requires that the estimate of \( \mu \) based on \( y \) be the same as the estimate of \( \mu \) based on \( x \); that is, \( T^*(x_1+a, \ldots, x_n+a) - a = T^*(y_1) - a = T(x) \).

(b) The formal structures for the problem involving \( x \) and the problem involving \( y \) are the same. They both concern a random sample of size \( n \) from a normal population and estimation of the mean of the population.

Thus, the formal invariance requires that \( T(x) = T^*(y) \) for all \( x \).

Combining with part (a),

\[
T(x_1+a, \ldots, x_n+a) - a = T^*(x_1+a, \ldots, x_n+a) - a = T(x_1, \ldots, x_n)
\]

i.e. \( T(x_1+a, \ldots, x_n+a) = T(x_1, \ldots, x_n) + a \), for all \( x_i \).

(c) \( \bar{W}(x_1, \ldots, x_n) = x \)

\[ \Rightarrow \bar{W}(x_1+a, \ldots, x_n+a) = \frac{1}{n} \sum (x_i+a) = \bar{x} + a = \bar{W}(x_1, \ldots, x_n) + a \]

So \( \bar{W}(x) \) is equivariant.

\( x_1, \ldots, x_n \sim \mathcal{N}(x \Theta) \)

\[ \Rightarrow x_0 \sim \mathcal{N}(x \Theta) \quad \text{where} \quad x_0 \sim \mathcal{N}(x \Theta) \]

\[ \Rightarrow E\bar{W} = \frac{1}{n} \sum (x_0+\Theta) = \Theta \], for all \( \Theta \).
2. (6.43)

(a) For location-scale family, if \( X \sim \frac{1}{\theta} f((x-\mu)/\theta) \), then

\[
Y = \delta_2 c(x) \sim \frac{1}{\sigma^2} f\left[ y - \mu + \sigma c \right]
\]

so for estimating \( \sigma^2 \), \( \hat{\delta}_2 c(o^2) = \hat{\sigma}^2 \).

An estimator of \( \sigma^2 \) is invariant w.r.t. \( G_1 \) if

\[
W(cX+\alpha, \ldots, cX_n+\alpha) = c^2 W(X_1, \ldots, X_n)
\]

\[
KS_2(cX+\alpha, \ldots, cX_n+\alpha) = \frac{1}{n-1} \sum (cX+\alpha - \bar{X}_c)^2 = \frac{1}{n-1} \sum (cX+\alpha - c\bar{X} + \alpha)^2
\]

\[
= c^2 \frac{1}{n-1} \sum (X_i - \bar{X})^2 = c^2 KS_2(X_1, \ldots, X_n)
\]

So \( KS^2 \) is invariant w.r.t. \( G_1 \).

For invariance w.r.t. \( G_2 \), just take \( c=1 \) above.

For invariance w.r.t. \( G_3 \), just set \( c=0 \) above.

So \( KS^2 \) is invariant w.r.t. \( G_1, G_2, G_3 \).

(b).

\[
W(X_1, \ldots, X_n) = \phi(\frac{X_1}{\theta}) \hat{\sigma}^2
\]

1. If \( W \) is invariant w.r.t. \( G_2 \),

\[
W(cX+\alpha, \ldots, cX_n+\alpha) = \phi(\frac{X+\alpha}{\theta}) \hat{\sigma}^2
\]

If \( \theta = 1, X = 0 \) this implies \( \phi(a) = \phi(0) \) for all \( a \),

i.e., \( \phi \) must be constant.

On the other hand, if \( \phi \) is constant, \( W \) is invariant
by part (a).

So \( W \) is invariant if and only if \( \phi \) is constant.

2. Since \( G_2 \) is a subgroup of \( G_1 \), invariance w.r.t. \( G_1 \)
also requires \( \phi \) to be constant.

\[
W(X_1, \ldots, X_n) = \phi(\frac{\bar{X}}{\theta}) \hat{\sigma}^2 = \phi(\frac{\bar{X}}{\theta}). c^2 \hat{\sigma}^2 = c^2 W(X_1, \ldots, X_n)
\]
3. (7.40) 
\[ L(x|\theta) = \prod_{i=1}^{n} p^{x_i} (1-p)^{1-x_i} \]
\[ \Rightarrow \log L(x|\theta) = \left( \frac{x_i}{\theta} \right) \log p + \left( \frac{n-x_i}{1-\theta} \right) \log (1-p) \]
\[ \frac{\partial \log L}{\partial \theta} = \frac{x_i}{p} - \frac{n-x_i}{1-p} = \frac{nx_i - np + p}{p(1-p)(x-p)} \]

By Corollary 7.3.15, \( \bar{x} \) is the UMVUE of \( p \) and attains the CRLB.

4. (7.41). \( \bar{x} \sim \text{N}(\mu, \sigma^2) \)

(a). \( E(x|\bar{x}) = \frac{\sigma^2}{\bar{x}} \cdot E\bar{x} = M(x|\bar{x}) = \mu \) if \( \bar{x} \sigma = 1 \).

(b). \( \text{Var}(x|\bar{x}) = \frac{\sigma^2}{\bar{x}^2} \cdot \text{Var}(\bar{x}) = \sigma^2 \cdot \left( \frac{\sigma^2}{\bar{x}} \right) \)

So we need to minimize \( \bar{x} \sigma^2 \) with constraint \( \bar{x} \sigma = 1 \).

Use Lagrange multipliers, we have

\[ L = \frac{\sigma^2}{\bar{x}} - \lambda (\bar{x} \sigma - 1) \]
\[ \Rightarrow \frac{\partial L}{\partial x} = 2\sigma \bar{x} - \lambda = 0 \]
\[ \Rightarrow x = \frac{\lambda}{2} \]
\[ \frac{\partial L}{\partial \lambda} = 2\sigma \bar{x} - 1 = 0 \]

So \( \bar{x} = \frac{\lambda}{2} \cdot \bar{x} = \frac{\sigma^2}{\bar{x}} \cdot \bar{x} \)

has the minimum variance among all unbiased estimators of this form.
5. (7.42). \( Ew_i = 0, \ Var(w_i) = \sigma^2, \ \text{Cov}(w_i, w_j) = 0 \text{ if } i \neq j. \)

(a). \( \mathbb{E}(\frac{\partial}{\partial \alpha} \mathbb{E}(w_i)) = \alpha (\text{Var}(w_i)) \Rightarrow \frac{\alpha}{\text{Var}(w_i)} = 0. \)

\[ \text{Var}(\mathbb{E}(w_i)) = \frac{\sigma^2}{\text{Var}(w_i)} = \frac{\sigma^2}{\alpha}, \text{Var}(w_i) = \frac{\sigma^2}{\text{Var}(w_i)} \]

So we need to minimize \( \frac{\sigma^2}{\text{Var}(w_i)} \) with constraints \( \sum \alpha_i = 1, \)

using Lagrange multipliers,

\[ L = \frac{\sigma^2}{\text{Var}(w_i)} - \lambda (\sum \alpha_i - 1) \]

\[ \Rightarrow \frac{\partial L}{\partial \alpha_i} = \frac{2 \alpha_i \sigma^2}{\sum \alpha_i} - \lambda = 0 \Rightarrow \left\{ \begin{array}{l}
\alpha_i = \frac{2 \sigma^2}{\lambda \sum \alpha_i} \\
\sum \alpha_i = 1
\end{array} \right. \Rightarrow \left\{ \begin{array}{l}
\lambda = \frac{2 \sigma^2}{\sum \alpha_i} \\
\alpha_i = \frac{\sigma^2}{\lambda \sum \alpha_i}
\end{array} \right. \]

\[ \Rightarrow w^* = \frac{\sum w_i \sigma^2}{\sum \sum \alpha_i} = \frac{\sum w_i \sigma^2}{\sum \lambda \sum \alpha_i} \text{ has minimum variance.} \]

(b). \( \text{Var}(w) = \frac{\sigma^2}{\text{Var}(w_i)} \)

\[ \Rightarrow \text{Var}(w^*) = \frac{\sigma^2}{\text{Var}(w_i)} \left( \frac{\sum \alpha_i}{\sum \sum \alpha_i} \right) = \frac{1}{\sum \sum \alpha_i} \]

6. (7.44). \( X_i \sim \text{iid } N(\theta, 1) \)

\[ \Rightarrow \sum X_i \text{ is complete sufficient statistic.} \]

\( X^2 - \bar{X}/n \) is a function of \( X_i \). By Theorem 7.3.23,

\( X^2 - \bar{X}/n \) is the unique best unbiased estimator of

its expectation \( E(X^2) = \text{Var}(X) + (E(X))^2 = \frac{1}{n} + \sigma^2 \Rightarrow \text{Var}(X) = \sigma^2 \)

\( \text{Var}(X^2 - \bar{X}/n) = \text{Var}(X^2) - (E(X))^2 = E(X^4) - (E(X))^2 \)

\( \bar{X} \sim N(\theta, 1/n) \)

\[ E(X^4) = \text{E}(X^2(X - \theta + \theta)) = \text{E}(x^2 - \theta) + \theta \text{E}(X^2) \]

\[ \text{E}(X^2(X - \theta)) = \frac{1}{n} \text{E}(\text{E}(X^2)) = \frac{\theta}{n} (\theta^2 + 1) \]

(by Steen's Lemma)

\( \theta \text{E}(X^3) = \theta \text{E}(X^2(X - \theta)) + \theta^2 \text{E}(X^2) = \frac{\theta}{n} (\theta^2 + 1) \)

\[ = \frac{\theta^4}{n} + \frac{\theta^2}{n} \]

\( \Rightarrow \text{Var}(X^4) = \frac{\theta^4}{n} + \frac{\theta^2}{n} + \sigma^4 + \theta^2 \sigma^2 \Rightarrow \text{Var}(X^2 - \bar{X}/n) = \frac{2 \sigma^2}{n} + \frac{\theta^2}{n} > \frac{4 \sigma^2}{n} \]

It's easy to compute \( CRB = -\frac{\theta^2}{nE(\text{Var}(X^2 - \bar{X}/n))} = \frac{4 \sigma^2}{n}. \)