Climatological analysis of tornado report counts using a hierarchical Bayesian spatiotemporal model

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[1] Recent evidence suggests that tornado report counts for monthly or longer periods may be correlated to climate indices in space and time. However, the climatological analysis of tornado reports is complicated by reporting errors, by the non-Gaussian nature of count data and rare events, and by the presence of spatial and temporal correlation. Typically, such factors have not been formally included in the underlying statistical model used for inferential decisions. We present an analysis in which a statistical model is constructed so that the aforementioned characteristics of tornado reports are explicitly accommodated in the model structure. In particular, a hierarchical Bayesian framework is considered, in which the various complicated structures are considered in a series of conditional models, formally linked by basic probability rules. This formalism allows one to evaluate characteristics of the spatiotemporal variability of an underlying (unobservable) tornado count process given the noisy observations (tornado reports). In addition, one can consider the effects of exogenous climate processes on this tornado count process. We find that an index of El Niño activity is significantly associated with tornado reports over the continental U.S. and that there is substantial regional variability in this relationship. In addition, we find evidence of temporal trends in tornado counts, with spatial variation in the magnitude and sign of the trend. INDEX TERMS: 1620 Global Change: Climate dynamics (3309); 3210 Mathematical Geophysics: Modeling; 3399 Meteorology and Atmospheric Dynamics: General or miscellaneous; KEYWORDS: Bayesian, hierarchical, Poisson, uncertainty, spatiotemporal


1. Introduction

[2] Interest in tornado climatology stems from the destructive potential of tornadoes. Clearly, the loss of human life caused by tornadoes is reason enough to obtain greater understanding. In addition, the estimated cost of damage caused by tornadoes is close to one billion per year in 1999 U.S. dollars [Brooks and Doswell, 2001] (see also R. A. Pielke Jr., Extreme Weather Sourcebook 2001 Edition, http://sciencepolicy.colorado.edu/sourcebook), even though most tornadoes occur in rural areas. It is important to gain a fundamental understanding of the climatological behavior of such phenomena in order to develop mitigation strategies for interests ranging from realistic assessment of tornado variability under different climate regimes to seasonal prediction on a regional scale. Because physical understanding of climate influences on the meteorological phenomena at the tornado scale is limited, it is useful to approach this problem from a statistical perspective.

1.1. Summary of Recent Literature

[3] Climatological analysis of tornado counts is complicated by the quality of tornado reports archived by the National Weather Service (NWS). As summarized by Doswell and Burgess [1988], most reports come from relatively untrained witnesses, with only a few events prior to the late 1970s subject to on-site inspection by trained specialists. Thus there are two primary limitations of such a database: reliance on human observation and a variable rating procedure. That is, the extent to which this database represents the true occurrence of tornadoes rests entirely on the willingness of humans to look for and report tornadoes and on the ability of humans to assign accurate assessments of tornado strength. It is very likely that some tornadoes are not witnessed, especially those that occur in rural areas and/or at night, while others do not cause damage. Thus the analysis discussed herein constitutes an examination of tornado reports rather than tornado occurrence.
[4] An overview of statistical characteristics of the severe weather report archive maintained by the NWS was provided by Brooks [1998]. He found that severe thunderstorm report intensities have increased dramatically since 1975. Tornado reports were found to exhibit similar changes over time, although regional differences were noted. For example, Florida and the front range counties of Colorado showed different reporting trends than other regions of the U.S.

[5] An approach for estimating the daily climatological probability (i.e., average frequency over a long time period) of F2–F5 tornadoes (where the “F” stands for the “Fujita scale,” a rating system for tornado damage, in which F0 represents “minimal damage” and F5 represents “complete destruction”) is proposed by Concannon et al. [2000]. Their analysis was based on a more comprehensive database of F2–F5 tornadoes compiled by Grazulis [1993] for the period 1921–1995. They found that (1) the maximum number of tornado days per century occurs in an L-shaped pattern, from southeastern Mississippi, to southern Oklahoma, to southwestern Iowa; (2) fluctuations as large as 25% of the climatological frequency have occurred, but a national trend is not evident, although it was noted that slightly more tornado days occurred in Oklahoma and Kansas during the early rather than late portion of the record; (3) the annual peak frequency exhibits less interannual variability at points in the central rather than eastern U.S.; and (4) peak frequency occurs in March and April in the southeast U.S. and progresses from Texas to Canada from April to July. Thus, their results support the presumption that climate fluctuations impact tornado report frequency.

[6] Bruening et al. [2002] apply the analysis procedure of Concannon et al. [2000] to F0–F5 tornado reports in order to examine year-to-year variations of tornado counts. They documented a clearly increasing trend in national tornado report counts of ~14 tornadoes per year, based on simple linear regression results. These authors also identified significant interannual and intraannual variability in tornado reports.

[7] Monfredo [1999] examined the association between tornado reports in the midsouth and southern plains of the U.S. and the Southern Oscillation index (SOI) using reports from the NWS log. Specifically, he used dew point temperature and precipitation to define two regions that had similar meteorological characteristics. The regions were labeled (1) southern low plains (SLP), which included Missouri, Arkansas, the portion of Kansas east of the 100th meridian, all of Oklahoma except the panhandle, and north central Texas, and (2) midsouth (MSO), which contained Kentucky, Tennessee, and the northern halves of both Alabama and Mississippi. He considered only F2–F5 tornadoes from February to July (his “tornado season”) of each year. SOI was accumulated over the canonical El Niño period of March–February, preceding the tornado season. Spearman’s rank correlation test showed significantly lower counts of F2–F5 tornado reports during El Niño and significantly higher counts during La Niña in both the SLP and MSO regions. The time lag between canonical El Niño and tornado seasons somewhat obscures the interpretation of this result, but Monfredo’s [1998] previous work suggests a physical rationale for such findings.

[8] Marzban and Schafer [2001] provided the most comprehensive analysis to date relating tornado counts and El Niño. They noted that recent studies presented at conferences on this subject showed conflicting results. A concern was that in each of these studies, El Niño was based on indices that differed in duration, underlying variables, timing with respect to the tornado season, and the indicator of anomalous behavior. In order to mitigate ambiguity of this type, they averaged equatorial Pacific sea surface temperatures (SST) in four zones and examined the monthly contemporaneous correlation between these SST indices with both the number of tornadoes and the number of tornado days that occur nationally and within certain regions of the U.S. They considered data from the NWS log for the years 1950–1998, and they found increasing trends in national counts of all (F0–F5) tornado reports related to all SST regions and a decreasing trend in national counts of F2–F5 tornado reports. They speculated that the increasing trend in F0–F5 tornado counts was due to increasing population and improvements in detecting tornadoes and that the decreasing trend of F2–F5 counts may have been related to a procedural change in the manner by which F-scale ratings were assigned.

[9] Marzban and Schafer [2001] used Kendall’s $\tau$ non-parametric measure of association to examine the relationship between tornado activity and the various SST indices. For tornado activity, they considered separately total report counts as well as number of days with tornadoes. They considered two intensity levels, F0–F5 ("all tornadoes") as well as F2–F5 ("strong and violent"). They found relatively weak correlations, in general, after removing seasonal effects. However, they did find statistically significant correlations between SST and tornado activity, with the level of significance varying with location over the U.S. and spatial region corresponding to the SST index. In particular, the most significant correlation ($p$ value = 0.0018) was between SST in the eastern equatorial Pacific and the number of tornadic days in the northeast U.S. Furthermore, the SST in the central equatorial Pacific was found to be significantly correlated with strong and violent tornado counts (and tornado days) in the northeast U.S., citing results from Hoerling et al. [1997] as physical evidence in support of this result. In particular, time-averaged upper-level wind fields show that the jet stream over the northeastern U.S. is stronger when the equatorial Pacific SSTs are cooler than normal. The central and southeast U.S. showed less significant associations in general. They concluded that tornado report frequency and equatorial Pacific SST are weakly negatively correlated.

1.2. Study Motivation

[10] The review of recent literature presented in section 1.1 suggests that statistical analyses of tornado climatology should account for: (1) observational uncertainty present in tornado report counts; (2) spatial variability in the long-term frequency of tornado counts; (3) temporal trends that may vary with space; and (4) exogenous climatological factors, primarily tropical Pacific SST, that are associated with tornado reports, with the potential for such effects to vary spatially.

[11] Indeed, the climatological analysis of tornado reports is not only complicated by reporting errors but by the non-
Gaussian nature of count data and rare events and by the presence of spatial and temporal correlation. The analyses to date have estimated sample statistics, such as sample mean frequency or sample correlation between tornado counts and climate indices. Hypothesis tests have often been applied to these estimates of sample statistics. In most cases, either an underlying assumption is made that the observations are independent, or the observations are grouped into relatively large geographical regions, potentially smearing the signal. Although some studies have accounted for some of the complicating factors mentioned above (e.g., preprocessing the data to remove spatial and temporal correlation), all of these factors have not been formally included in the underlying statistical model used for inferential analysis.

[12] We take an alternative approach, in which a statistical model is constructed so that the aforementioned characteristics of tornado reports are explicitly accommodated in the model structure. In particular, a hierarchical Bayesian framework is considered, in which the various complicated structures are considered in a series of conditional models, formally linked by basic probability rules. This formalism allows one to evaluate characteristics of the spatiotemporal variability of the underlying (unobservable) tornado count process given the noisy observations. Such a model can easily incorporate spatially varying climatological effects (e.g., overall spatial mean and trend) as well as spatially varying effects of temporally varying exogenous variables (e.g., El Niño) on the underlying tornado count process.

2. Data

[13] Tornado reports submitted to the NWS from 1950 to 1995 are archived at the National Oceanic and Atmospheric Administration Storm Prediction Center. We have extracted all reports from 1953 to 1995 for our analysis, since tornado reports prior to 1953 were very infrequent. This was also the first year of issuance of severe weather watches by the NWS. A short list of information contained in each report includes the time and location of the initial and final observation of the tornado, estimated tornado path width, estimated tornado intensity, estimated dollar amount of damage produced by the tornado, and other forms of severe weather observed in the immediate vicinity of the tornado. The accuracy of estimated observations, such as tornado intensity or cost of damage, are open to debate since they rely upon interpretation by human observers with varying levels of expertise in assessing these characteristics [Doswell and Burgess, 1988]. In this study we examine the initial position of each tornado report, which is considered the most reliable entry.

[14] We overlaid a grid (2400 km x 1700 km) of 50-km boxes on the continental U.S., with the upper-left grid box centered at ~45.5°N, 105°W and with 54 grid boxes in the east-west direction and 34 in the north-south direction. Because tornado reports were very infrequent within the Rocky Mountains and along the West Coast, we considered only tornado report counts east of 105°W. For every year we tallied the number of initial observations of an F2–F5 tornado that occurred within each box (hereinafter referred to as tornado report counts), resulting in a time series of length 43 years for each of the 1836 grid boxes. Figure 1 shows the total counts observed in each grid box over the 43 years. Note that there are regions with no observations (e.g., oceanic areas). Furthermore, the NWS database does not include tornado report counts over Canada. In general, there are a significant number of zero counts over the domain (i.e., ~90% of the observations are zero).

[15] In this study we considered the spatiotemporal distribution of the tornado report counts and the relationship of these counts to an index of the El Niño/Southern Oscillation (ENSO) phenomenon, as given by the Niño3.4 index [e.g., Trenberth, 1997]. These data were obtained as monthly time series from the National Centers for Environmental Prediction Climate Prediction Center web site. Yearly average (mean) values of the index were used in the study so as to correspond to the yearly tornado report counts.

3. Methodology

[16] Tornado counts are distinctly non-Gaussian (i.e., counts are discrete), have significant sampling and observational variability, contain many zeros, and exhibit relatively complicated spatiotemporal dependence. These issues make traditional regression methods intractable. In contrast, Bayesian hierarchical models attempt to decompose the observed data into a series of conditional models, all linked together formally through basic probability relationships. In this way, one can account for the observational process error, the “true” process variability (e.g., the variation in true counts, which can never be observed), and parameter uncertainty (e.g., the strength of correlation among parameters).

[17] The basic hierarchical model is then written in three primary stages: (1) data model ([data|process, parameters]); (2) process model ([process|parameters]); and (3) parameter model ([parameters]), where the brackets refer to the “probability distribution” and the vertical line indicates a conditional distribution. Thus the fundamental idea is to approach the complex statistical modeling problem by breaking it into pieces [e.g., Berliner, 1996; Wikle et al., 1998].

[18] The first stage is concerned with the observational process, or “data model,” which specifies the distribution of the data (e.g., observational counts of tornadoes) given the process of interest (e.g., the true tornado count over some geographical area) and parameters that describe the data model. The second stage then describes the process, conditional on other parameters. For example, this might be a “regression” model, relating tornado occurrence to climate indices, with parameters describing the strength of the association. Finally, the last stage accounts for the uncertainty in the parameters by assigning them distributions. For example, the effect of a climate index could be different at different geographical locations. Thus we might allow the regression parameters to be spatially varying and assign them a distribution that includes a spatial correlation.

[19] Ultimately, our interest is with the distribution of the process and parameters, updated by the data. This so-called “posterior” distribution is obtained via Bayes’s theorem [e.g., Gelman et al., 1995]. Specifically, the distribution [process, parameters|data] is proportional to the product of the data model, process model, and parameter model from the three stages listed above:

\[
\text{process, parameters|data} \propto \text{data|process, parameters} \cdot \text{process|parameters} \cdot \text{parameters}. \tag{1}
\]
This formula serves as the basis for estimation in Bayesian hierarchical models. Although simple in principle, there are several critical issues. First, the specification of the component distributions in equation (1) can be challenging. However, this is the case with any statistical model, and the specification of each stage requires the modeler to address issues about uncertainty directly (rather than implicitly, as in many traditional statistical models). Much of the uncertainty quantification is subjective (e.g., choice of distributions, parameterizations, etc.), but the quantification of such subjective judgment is, in fact, the strength of the Bayesian approach in that it provides a coherent probabilistic framework in which to incorporate scientific reasoning, judgment, and experience explicitly in the model (see Carlin and Louis [2000] for an overview of issues related to Bayesian modeling).

[20] The second critical issue concerns the calculation of the posterior distribution of equation (1). In principle, this is simple, but in practice, it can be very difficult to obtain the normalizing constant for the posterior distribution due to the complexity and high dimensionality of the model. However, although we cannot usually compute the posterior directly, we can simulate from it using Markov Chain Monte Carlo (MCMC) computational procedures. Such procedures for Bayesian computation have become common in recent years (e.g., see Gilks et al. [1996], Robert

Figure 1. (top) Log counts of F2–F5 tornadoes summed over the years 1953–1995. (bottom) Time series of F2–F5 tornado counts for each year summed over the spatial domain shown in Figure 1 (top).
3.1. Hierarchical Model for Tornado Counts

Poison regression is often desired when the response outcomes are counts, especially in cases where large counts are relatively rare. In addition, one wishes to relate the expected mean response (i.e., the Poisson intensities) to a set of covariates. The general class of models for such a process are known as generalized linear models (see McCullagh and Nelder [1989] for an overview and Yan et al. [2002] for an illustrative meteorological example). This procedure can be extended to accommodate random effects as well as fixed effects, in which case the class of models is known as generalized linear mixed models (e.g., see McCulloch and Searle [2001] for an overview). In recent years, investigators have considered complicated random effects such as those induced by spatial random fields. Although there have been some classical implementations of such models (e.g., Heagerty and Lele, 1998), typically, their added complexity has led researchers to consider hierarchical Bayesian implementations (e.g., Diggle et al., 1998; Royle et al., 2002; Best et al., 2002; Christensen and Waagepetersen, 2002; Crainiceanu et al., 2003; Wikel, 2002). In the case of tornado count data we would like to relate the mean of the underlying Poisson count process to a spatially varying temporal trend as well as a spatially varying ENSO effect, where the regression coefficients are themselves spatial random processes. However, examination of the data suggests that there are too many zeros to justify a Poisson model for counts. Rather, we consider a so-called zero-inflated Poisson model, in which there is a probability $p$ that the observation is from a Poisson random variable and a probability 1-$p$ that the only possible observation is 0 [e.g., Lambert, 1992; Hall, 2000; Nair et al., 2001]. The hierarchical model is described in detail in sections 3.1.1 – 3.1.4.

3.1.1. Data Model

Let $Y(s_i,t)$ be the tornado counts in the geographical region (i.e., grid box) indexed by $s_i$, $i = 1, \ldots, n$ at times $t = 1, \ldots, T$ (representing years 1953,..., 1995). Given that there are too many zero counts to justify a Poisson model, we utilize a zero-inflated Poisson data model [e.g., Lambert, 1992] given by

$$Y(s_i,t)|\lambda(s_i,t),p(s_i,t) \sim \begin{cases} \text{Poi}(\lambda(s_i,t)) & \text{with probability } p(s_i,t); \\ 0 & \text{with probability } 1-p(s_i,t) \end{cases}$$

for all $i$ and $t$. That is, conditional on the Poisson intensity (or mean) $\lambda(s_i,t)$, which is a spatiotemporal process, and on a spatiotemporal probability process $p(s_i,t)$, the tornado counts are either 0 or follow a Poisson distribution. Thus it is a mixture distribution of a point mass at 0 and a Poisson distribution. The observations are conditionally independent. This does not suggest that the counts at two locations/times (e.g., $Y(s_i,t)$ and $Y(s_j,\tau)$) are independent but rather that their spatiotemporal dependence is due to the underlying intensity process, $\lambda$, and probability process, $p$, which are modeled at the next stage. We say that the $Y$ process is dependent marginally but is independent conditionally.

3.1.2. Process Models

The process models describe the underlying spatiotemporal processes, $\lambda(s_i,t)$ and $p(s_i,t)$. First, we consider the $\lambda$ process, as it is the process of primary interest meteorologically.

In the present case we believe that the underlying spatiotemporal variability of the $\lambda$ process is partially explained by a linear temporal trend and by an ENSO effect. Furthermore, there are spatiotemporal random effects that account for the additional variability. That is,

$$\log(\lambda(s_i,t)) = \beta_1(s_i)x_{1t} + \beta_2(s_i)x_{2t} + \eta(s_i,t) + \epsilon(s_i,t),$$

where $\eta_k \equiv (\eta(s_1,t), \ldots, \eta(s_n,t))'$ is a spatially correlated (but temporally uncorrelated) random process with mean $\delta_0$, correlation matrix $R(\theta_0)$, and variance $\sigma_{\eta k}^2$, $\epsilon(s_i,t) \sim iid N(0,\sigma_{\epsilon}^2)$, and where $\beta_k(s_i)$ is a regression parameter at location $s_i$ for the $k$th covariate, $x_{kt}, k = 1, 2$. That is, the log of the Poisson intensity is modeled by a linear regression with normal errors such that the regression coefficients are spatially varying so that each site has its own impact from the given covariate. In this analysis, covariates include a year index ($x_{1t} = \{1, \ldots, 43\}$) and the ENSO climate index ($x_{2t}$), although, as described in section 3.2, these covariates are centered and standardized in the implementation. Thus $\beta_1$ represents the temporal trend, and $\beta_2$ represents the ENSO effect. The important point here is that these coefficients are spatially varying as well. The error process represented by the term $\epsilon(s_i,t)$ accounts for spatiotemporal random effects representing unobserved or unknown climate effects that are spatially correlated, whereas $\epsilon(s_i,t)$ represents such random effects that are not spatially correlated (e.g., see Best et al. [2002] for similar formulation in the spatial context). That is, these processes account for variability in the spatiotemporal Poisson intensity process not directly modeled by the aforementioned covariates. For example, these noise processes can partially account for geographical differences caused by sampling biases (e.g., demographic biases) and/or by unspecified climatological and meteorological factors.

The zero-inflated Poisson probability process $p(s_i,t)$ is traditionally modeled on the logit scale [Lambert, 1992; Hall, 2000; Nair et al., 2001]. In our case,

$$\logit(p(s_i,t)) = \nu(s_i) + \gamma(s_i,t),$$

where logit($z$) $\equiv \exp(z)/(1 + \exp(z))$ is the logit function, $\nu(s_i)$ is a spatial random effect (specified at the next level of the hierarchy) and $\gamma(s_i,t) \sim iid N(0,\sigma_{\gamma}^2)$ is an uncorrelated spatiotemporal error process. In this case we know less about the probability process, $p(s_i,t)$, a priori than we do about the $\lambda(s_i,t)$ process. However, we believe that the underlying process that forces counts to deviate from a true Poisson process is likely to be spatial rather than temporal in nature. For example, areas over water and over regions for which we have no tornado reports (e.g., Canada) obviously have too many zeros. We will account for this in the mean structure for the spatially varying $\nu$ process, as shown in section 3.1.4.

3.1.3. Parameter Models for the Poisson Mean Process, $\lambda$

At this stage of the hierarchical model the parameters are given distributions. For example, it is our prior belief,
based on the published literature (see the literature review in section 1.1), that the $\beta$s in the process model vary spatially and that they should exhibit spatial coherence. That is,

$$\beta_k | \theta_k, \sigma_k^2 \sim N(0, \sigma_k^2 \mathbf{R}_k(\theta_k)), \text{ for } k = 1, 2$$

(5)

where $\beta_k$ is an $n \times 1$ vector of rasterized and spatially indexed coefficients corresponding to the $k$th covariate, $\sigma_k^2$ is the variance of this parameter process, and $\mathbf{R}_k(\theta_k)$ is a spatial correlation matrix that itself depends on the parameter $\theta_k$. Again, the important point here is that our prior belief is that there may be spatial structure in the regression coefficients. That is, if the regression coefficient in one grid box is high, we expect that the adjacent grid boxes will be more likely to show high coefficients as well. In this way, estimates of any given $\beta_k(s_i)$ can "borrow strength" from their neighbors, so the effective number of parameters actually estimated for $\beta_k(s_i), i = 1, \ldots, n$ is much less than $n$. However, the model must be flexible enough that if the data suggest that this is not the case, it can be reflected in the posterior analysis. Hence the spatial dependence of each $\beta$ is controlled by the parameter $\theta_k$, itself a random parameter with a specified distribution (see equation (15)).

[27] Similarly, as mentioned in section 3.1.2, the $\eta$ process is spatially correlated with prior distribution

$$\eta_k | \delta_k, \sigma_k^2, \theta_k \sim N(\delta_k, \sigma_k^2 \mathbf{R}(\theta_k)).$$

(6)

[28] In practice, the implementation of the hierarchical model in the case with parameter model equations (5) and (6) can be very difficult because of the high dimensionality of the spatial processes (recall that $n = 1836$ in our case, implying, for example, that equation (5) is a 1836-dimensional multivariate normal distribution and $\mathbf{R}_q(\theta_q)$ is $1836 \times 1836$. Alternatively, as shown by Wikle [2002], Bayesian computation of Poisson hierarchical spatial models is facilitated by considering spectral representations of the spatial processes. Specifically, we let $\beta_k = \Phi \alpha_k$, and $\eta_k = \Phi \xi_k$, where $\Phi$ is an $n \times n$ matrix of spectral basis functions. Thus we can rewrite equation (3) as

$$\log(\lambda_i) = \Phi \alpha_{1i} x_{1i} + \Phi \alpha_{2i} x_{2i} + \Phi \xi_i + \epsilon_i,$$

(7)

where $\Phi$, $x_{1i}$, and $x_{2i}$ are known and equations (5) and (6) become

$$\alpha_k | \theta_k, \sigma_k^2 \sim N(0, \sigma_k^2 \mathbf{D}(\theta_k)), \text{ for } k = 1, 2$$

(8)

$$\xi_i | \delta, \theta, \sigma_k^2 \sim N(\delta, \sigma_k^2 \mathbf{D}(\theta_k)), \text{ for all } i,$$

(9)

where $\mathbf{D}(\theta_k) \equiv \Phi^T \mathbf{R}(\theta_k) \Phi$ and $\mathbf{D}(\theta_q) \equiv \Phi^T \mathbf{R}(\theta_q) \Phi$.

[29] In principle, any orthogonal basis set can be chosen for $\Phi$. For example, possible choices for basis functions in these settings include empirical orthogonal functions [e.g., Obled and Creutin, 1986], wavelets [e.g., Wikle et al., 2001], and Fourier basis functions [e.g., Wikle, 2002]. The Fourier and wavelet modes are advantageous computationally when data are gridded (as in the present case) due to the computational efficiency of discrete transform algorithms (note that if $\Phi$ is a matrix of Fourier basis functions, then $\Phi \alpha$ is just the inverse discrete Fourier transform operation on the vector $\alpha$ and $\Phi^T \mathbf{y}$ is just the discrete Fourier transform (DFT) of $\mathbf{y}$). Furthermore, if the spatial process is stationary, then use of the Fourier basis set implies that the matrix $\mathbf{D}(\theta_k)$ is asymptotically diagonal [e.g., Shumway and Stoffer, 2000, section T3.12].

[30] In the present case we assume the correlation structure of these processes is isotropic and stationary and that they follow an exponential model ($r(h) = \exp(-\theta h)$, where $h$ is the Euclidean distance between sites) but allow the dependence parameter $\theta$ to be random (modeled at the next stage of the hierarchy). Thus, conditional on the dependence parameter, one can obtain an analytical expression for the matrix $\mathbf{D}(\theta_k)$, as shown by Wikle [2002]. That is, for the exponential correlation function corresponding to a spatial process in two dimensions the corresponding spectral density function at frequency $\omega = f(\omega) \propto 1/(\omega^2 + \omega^2)$ [see e.g., Stein, 1999, p. 49]. Thus if $\Phi$ are Fourier basis functions, then the prior structure for $\mathbf{D}(\theta)$ is assumed to be diagonal (this is true asymptotically and is thought to be a reasonable prior assumption here) and is defined completely by the spatial dependence parameter $\theta$. That is, this matrix is diagonal, with the diagonal elements corresponding to the spectral densities given by $f(\omega)$. In practice, one must make sure that the rasterization (that takes the two-dimensional spatial field into a vector) of the two-dimensional spectral density corresponds with the particular DFT and inverse DFT algorithms used in the implementation. In addition, one must account for edge effects (e.g., by zero padding) as is typical for DFT applications. Finally, we note that the class of covariance functions that can be considered in this way is much broader than that presented here. For instance, Wikle [2002] discusses the implementation of the Matern class of covariance functions, in which the inherent smoothness (i.e., differentiability) of the correlation function is parameterized as well as the spatial dependence.

[31] Typically, spectral transformations, such as the Fourier transform described above, can be truncated without substantial loss of information. In our case we do not truncate as we have little prior information to suggest which coefficients are needed, especially since we are not sure how much spatial dependence should be in the spatial random fields. In addition, the DFT algorithm is very efficient and requires the full-dimensional spatial field for implementation. We note, however, as shown below, that many of the $\alpha$ coefficients are very close to zero in the posterior mean.

[32] One might question the choice of a stationary isotropic spatial model when the spatial process covers such a broad geographical region as is the case here. This may be a strong concern for the $\lambda$ process, but we have little prior knowledge to suggest that the $\beta$ processes or the $\eta$ process are nonstationary. To account for spatial nonstationarity, one could use wavelet basis functions to model the spatial processes, as in the work of Wikle et al. [2001] and Nychka et al. [2002]. This will be explored elsewhere.

[33] The $\delta$ process has prior distribution given by

$$\delta | \sigma_{\delta 1}^2, \sigma_{\delta 2}^2 \sim N(0, \text{diag} [\sigma_{\delta 1}^2, \sigma_{\delta 2}^2, \mathbf{I}_{n-1}]),$$

(10)
Table 1. Prior and Posterior Summary of Univariate Model Parameters

<table>
<thead>
<tr>
<th>Parameter Mean</th>
<th>Prior Variance</th>
<th>Prior Parameters</th>
<th>Posterior Mean</th>
<th>Posterior Variance</th>
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<tr>
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<td>0.81</td>
<td>0.32</td>
</tr>
<tr>
<td>$\sigma^2_2$</td>
<td>0.2</td>
<td>$q_2 = 2.020, r_2 = 4.902$</td>
<td>0.12</td>
<td>0.68</td>
</tr>
<tr>
<td>$\sigma^2_3$</td>
<td>0.05</td>
<td>$q_3 = 2.0025, r_3 = 19.9501$</td>
<td>0.00073</td>
<td>$5.6 \times 10^{-7}$</td>
</tr>
<tr>
<td>$\sigma^2_4$</td>
<td>-0.1</td>
<td>$a_1 = -0.1, \sigma^2_5 = 10$</td>
<td>-0.0059</td>
<td>0.0016</td>
</tr>
<tr>
<td>$\sigma^2_5$</td>
<td>0.01</td>
<td>$a_2 = 0.01, \sigma^2_6 = 10$</td>
<td>0.00020</td>
<td>$4.06 \times 10^{-4}$</td>
</tr>
</tbody>
</table>

where $\text{diag}(Z')$ refers to a diagonal matrix with elements of the vector $z$ on the main diagonal and $1_{n-1}$ is an $n - 1$ dimensional vector of ones. The first component of the $\delta$-process has variance $\sigma^2_{\delta,1}$, and the other $n-1$ elements have homogeneous variance given by $\sigma^2_{\delta,2}$. The motivation for this is our prior experience that the first (mean) component of spectral random fields is often substantially larger than the remaining elements. We specify inverse gamma (IG) prior distributions for all of the variance components:

$$
\sigma^2_\delta \sim \text{IG}(q_\delta, r_\delta),
$$

(11)

$$
\sigma^2_i \sim \text{IG}(q_i, r_i),
$$

(12)

$$
\sigma^2_j \sim \text{IG}(q_j, r_j), j = 1, 2
$$

(13)

$$
\sigma^2_{\delta, j} \sim \text{IG}(q_{\delta, j}, r_{\delta, j}), j = 1, 2
$$

(14)

where the $q$ and $r$ hyperparameters are specified as given in Table 1.

Finally, we specify prior distributions for the spatial dependence parameters:

$$
\theta_k \sim U(u_1, u_2), k = 1, 2
$$

(15)

$$
\theta_k \sim U(u_1, u_2),
$$

(16)

where $U(u_1, u_2)$ refers to a uniform distribution between $u_1$ and $u_2$ and where the hyperparameters $u_1, u_2$ are specified as given in Table 1. Small values of $\theta$ correspond to strong spatial dependence, and large values correspond to weak spatial dependence. Also, we use a standardized distance unit for the exponential model, in which the distance between adjacent grid boxes (50 km) is considered one distance unit.


The spatial random process associated with the logit of $p(s; t)$ is modeled as

$$
v(s) \alpha_1, \alpha_2, \sigma^2_v \sim N(\alpha_1 z_1(s) + \alpha_2 z_2(s), \sigma^2_v),
$$

(17)

independently for all $i$ and where $z_1, z_2$ represent dummy variables related to “data-rich” and “data-poor” regions. Specifically, $z_1(s)$ takes the value of 1 if the grid box is in a “data-rich” region and zero otherwise; $z_2(s)$ takes the value of zero for data-sparse grid boxes and 1 otherwise. A grid box is assumed to be “data-sparse” if it is in a boundary region (e.g., water, Canada, western part of domain) for which we have no tornado observations over the 43 years of data coverage. We let the $a$ parameters have prior distributions

$$
a_j \sim N(\tilde{a}_j, \tilde{\sigma}^2_a)
$$

(18)

for $j = 1, 2$, where $\tilde{a}_j, \tilde{\sigma}^2_a$ are hyperparameters specified as given in Table 1. Similarly,

$$
\sigma^2_g \sim \text{IG}(q_g, r_g),
$$

(19)

where the hyperparameters $q_g, r_g$ are specified as given in Table 1. In addition, the error variance associated with the logit$(p(s; t))$ spatiotemporal error process, $\gamma(s; t)$, is given a prior

$$
\gamma(s; t) \sim \text{IG}(q_\gamma, r_\gamma),
$$

where the hyperparameters $q_\gamma, r_\gamma$ are specified as given in Table 1.

3.2. Model Implementation

Given the hierarchical representation presented in sections 3.1.1–3.1.4, one can then evaluate the posterior distribution of all of the processes and parameters given the observed counts:

$$
\left\{\lambda_1, \ldots, \lambda_T, \mathbf{p}_1, \ldots, \mathbf{p}_T, \alpha_1, \alpha_2, \xi_1, \ldots, \xi_T, v, a_1, a_2, \sigma^2_v, \sigma^2_g, \sigma^2_\delta, \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}, \mathbf{g}, \mathbf{h}, \mathbf{i}, \mathbf{j}, \mathbf{k}, \mathbf{l}, \mathbf{m}, \mathbf{n}, \mathbf{o}, \mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s}, \mathbf{t}, \mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z}\right\}
$$

$$
\times \left\{\prod_{t=1}^T \prod_{j=1}^n \left[\gamma(s; t) | \lambda(s; t), p(s; t)\right], \lambda(s; t) | \alpha_1, \alpha_2, \xi_1, \xi_2, \sigma^2_v, \sigma^2_g, \sigma^2_\delta, a_1, a_2, \theta_1, \theta_2, \theta_3\right\}
$$

$$
\times \left\{\prod_{t=1}^T \left[\xi_1 | \delta, \theta_1, \sigma^2_\xi \right], \left[\sigma^2_\xi | \theta_1, \delta \sigma^2_\delta, \sigma^2_{\delta, 2}\right], \sigma^2_v \left[\mathbf{a} | \mathbf{b}, \sigma^2_v \right], \sigma^2_g \left[\mathbf{a} | \mathbf{b}, \sigma^2_g \right], \sigma^2_\delta \left[\mathbf{a} | \mathbf{b}, \sigma^2_\delta \right], \mathbf{a} | \mathbf{b}, \sigma^2_v, \sigma^2_g, \sigma^2_\delta\right\}
$$

(20)

where $\mathbf{Y}$ represents all observed counts and where $\lambda_n$ and $\mathbf{p}_t$ are $n \times 1$ vectorizations of the $n$ spatial locations at time $t$ for the Poisson means and zero-inflated probabilities, respectively. One cannot evaluate this posterior distribution analytically and must resort to numerical simulation methods. In particular, MCMC methods can be used to generate samples from the posterior distribution. See Wikle et al. [1998], Berliner et al. [2000], and Wikle et al. [2001] for examples of MCMC applied to atmospheric science problems. For MCMC implementations of Poisson spatial models, see Diggle et al. [1998], Best et al. [2002], Christensen and Waagepetersen [2002], and Crainiceanu et al. [2003]. For MCMC implementation of spatiotemporal Poisson models, see Wikle [2001] and Wikle [2003]. In the present paper we utilize a special case of MCMC known as the Gibbs sampler. This is described in more detail in
Thus was modeled in the process, the mean was deferred to the following stage and rather than in the model for \( \log( \cdot ) \) simply taking the inverse Fourier transform of the conceptualize in physical space rather than in spectral space. Processes, respectively. Scientific interpretation is easier to the mean of the random effect at the next stage \([e.g., \text{Gilks and Roberts [2000]}, \text{Carlin and Louis [1999]}]\) for an overview of approaches). Note there was enough dependence to identify the associated variance components, with posterior means \( \sigma_1^2 = 0.06 \) and \( \sigma_2^2 = 0.23 \), yet the mixing was rather slow, as mentioned in section 3.2. The variance components for the \( \beta \) processes were relatively small by comparison to the sum of these noise components \([e.g., \sigma_1^2 = 0.06 \) and \( \sigma_2^2 = 0.03 \)]). However, we did not expect that the temporal trend and ENSO components would be the dominant components of variability in the tornado counts but rather that they would be important components. On the basis of the variance of these distributions and the posterior distributions shown in Figure 2, these variance components do represent a “significant” portion of the tornado count variability. The “significance” of the \( \beta \) estimates is discussed further in section 4.2.

Although the zero-inflated probability process is considered more as a “nuisance” process in this analysis, it is instructive to examine the associated parameters. First, we note from the comparison of the posterior means of \( \sigma_1^2 \) and \( \sigma_2^2 \) that the spatial process \( \nu \) is not particularly important. However, examination of the spatial map of the \( \nu \) process
shown in Figure 3 suggests that the probability that the count is Poisson tends to be >0.5 in regions of largest counts and <0.5 in the Appalachian corridor and edge regions (for which there are few nonzero observations). Note that a value of $\nu(s_j) > 0$ suggests that there is a >0.5 probability that the process follows a Poisson distribution and that a value of $\nu(s_j) < 0$ suggests that the probability is <0.5 that the process is Poisson. Thus the model is performing as expected in that it is increasing the probability of zeros in areas where observations are limited or the process cannot be nonzero. However, we note that the differences in probabilities implied by this $\nu$ process are not very large (e.g., the spatial variation in $p$ implied by $\nu$ ranges between 0.49 and 0.51).

More sophisticated models could be developed for the $\nu$ process, although such models are not considered here. We do note that the trend and ENSO covariates were considered in the logit of the $p$ process in a preliminary analysis. These results did not show a strong relationship and thus were not considered further.

4.2. Results Related to Tornado Climatology

[44] The parameters of most meteorological importance related to tornado climatology are $\beta_1$, $\beta_2$, and $\eta_t$. Recall that these are obtained directly from samples of $\alpha_1$, $\alpha_2$, and $\xi$, through the inverse DFT, respectively. For simplicity of interpretation we focus our results on the spatial domain rather than on the spectral. It is interesting to note, however, that the majority of the spectral coefficients for the $\alpha$
processes are near zero (i.e., 80% of the posterior mean for $\alpha_1$ coefficients are between $-0.02$ and $0.02$; 88% of the posterior means for $\alpha_2$ are within this interval). This is not surprising as the Fourier transform does a good job of dimension reduction when the spatial correlation is relatively large, which is the case for $\alpha_1$ and $\alpha_2$, as discussed in section 4.1. In contrast, only 4% of the $\xi$ posterior mean coefficients are between $-0.02$ and $0.02$, reflecting the fact that there is very little spatial correlation in the $\eta$ random component, as discussed in section 4.1. That is, the $\xi$ process is not much different than a white noise process (flat spectrum).

Figure 5. (top) Posterior mean and (bottom) posterior standard deviation for $\beta_1$ (linear temporal trend) parameters.

[45] In general, posterior means of $\delta_0$ (i.e., the inverse Fourier transform of $\delta$) shown in Figure 4 reflect the spatial pattern within the raw counts. It shows relative minima in Missouri, in the western plains states, and in the Appalachian corridor. In addition, there are relative maxima in northeast Texas, Oklahoma, eastern Kansas, Iowa, Indiana, Alabama, Mississippi, and, perhaps surprisingly, western Connecticut. The spatial pattern of this process exhibits much more detail than the results of Concannon et al. [2000] that were produced from the so-called NWS-smoothed log. The similarities between raw counts and other smoothed plots lend credibility to the model results.
The maps of posterior means and posterior standard deviations for the $\beta$ parameters are given in Figures 5–6. If one considers the posterior distribution of the $\beta$s, the spatial regions corresponding to large magnitudes in the posterior mean in Figures 5 and 6 have 95% credible regions that are unlikely to cover zero, as shown in Figure 7. Thus one can say that these regions have less than a 5% probability of showing no response and thus are “significant” in a Bayesian (credible) sense. In addition, comparison with the total report counts over all years shown in Figure 1 shows that, as one would expect, the largest standard deviations correspond to areas of little or no data. This can be seen more clearly in Figure 8, which shows a scatterplot of the posterior standard deviations for $\beta_2(s_i)$ versus the total raw tornado counts for each grid box summed over all years.

It has been consistently reported that national counts of F2–F5 tornadoes have either remained nearly constant or slightly decreased over the period analyzed herein [Marzban and Schaefer, 2001]. The spatial field for posterior means of $\beta_1$ (the linear trend coefficients) shown in Figure 5 exhibits substantial spatial variability. Relatively large positive values of $\beta_1$ (indicating positive trends over time) are found near urban areas, such as along the eastern seaboard and...
near Denver, Colorado, whereas a large region of negative values (indicative of negative temporal trends) are located in the central U.S. and in the Florida panhandle region, where the spatial mean is relatively large. The large-scale spatial structure in this map suggests a southwest to northeast orientation to the positive and negative estimates. These results do not necessarily contradict previous results. It is possible that a large increase of tornado reports in regions where tornadoes have been and still are infrequent may be offset by decreases over the central U.S., where tornado frequency is much higher. However, it is clear that a national trend should not be considered representative of local trend.

[48] Posterior means of $\beta_2$ (Figure 6) indicate that dependence does exist between equatorial Pacific SST anomalies (Niño 3.4) and tornado report frequency in the U.S. An expansive area of negative values in the southeast U.S. suggests a reduced (increased) risk of F2–F5 tornadoes during El Niño (La Niña) in this region, while the opposite relationship is suggested for the western plains states. We note that one has to be careful in interpreting the maps near the boundaries due to edge effects and limited data. This result confirms findings from Marzban and Schaefer [2001] and Monfredo [1999] and establishes that the spatial pattern of this association is very coherent, yet regionally variable.

[49] Smaller regions with small positive and negative values throughout the central U.S. are inconsistent with results from Monfredo [1999]. Monfredo [1999] has reported significant seasonal lag correlation for F2–F5 tornado reports that were accumulated over Missouri, Arkansas, eastern portions of Kansas and Oklahoma, and northern Texas. A similar inconsistency with results from Marzban and Schaefer [2001] is found in the northeast U.S. Our approach differs from both studies in that, by using annual tornado counts, we have simultaneously examined effects of both seasonal lag and contemporaneous dependence. Further research is needed to determine how our results are influenced separately by lag and contemporaneous dependence at less than annual timescales in order to fully explain these contradictions.

[50] Positive association in the western plains states is a unique result. Previous studies have not considered the relationship between equatorial Pacific SST and tornado reports in this region. However, positive seasonal lag correlation between equatorial Pacific SST and summertime precipitation has been identified in this region [Ropelewski and Halpert, 1996; Ting and Wang, 1997]. Since nearly all summertime precipitation in this region is supplied by thunderstorms, increased precipitation implies an increase in either thunderstorm frequency or precipitation efficiency of thunderstorms. If it is shown that thunderstorm frequency increases, it might be reasonable to find an increase of F2–F5 tornado frequency as well since most F2–F5 tornadoes are spawned from thunderstorms.

5. Discussion

[51] The statistical model applied herein to tornado report counts is an exploratory tool. Confidence in the validity of our results leads us to pursue further our analysis in the form of physical diagnoses, with the goal of determining causal links between equatorial SST and tornado reports. Much has been learned in the past 5 years about mesoscale weather conditions that support tornadic thunderstorms. The question we are led to ask is, How might slowly varying climate forces create situations in which such mesoscale conditions repeatedly form or fail to form? This will require analyses of mean and transient flow during El Niño and La Niña events. Already, it is known that the wintertime jet stream over the northeastern U.S. is sometimes stronger during La Niña than during El Niño [Hoerling et al., 1997], which, as noted by Marzban and Schaefer [2001], could provide favorable conditions for tornadic thunderstorms should the anomaly persist into spring and summer months. It is important to follow up this finding with more detailed analysis. That is, it is necessary to determine what role, if any, transient synoptic-scale systems play in producing environments supportive of tornadic thunderstorms. In addition, the immediate environment of tornadic thunderstorms should be examined to determine if the structure of vertical wind shear and thermal stability resemble that of tornadoes elsewhere in the U.S.
It is possible that land surface memory may play a role in the southeast U.S., where negative association is evident between F2–F5 tornado report counts and equatorial Pacific SST anomalies. It is known that wintertime conditions in the southeast are sometimes wetter and stormier than normal during El Niño [Ropelewski and Halpert, 1996]. This suggests that springtime surface conditions would be more moist than average, which would affect parameters sensitive to thunderstorm development such as available buoyant energy and the height of the lifted condensation level. Monfredo [1998] has found that temperature inversions immediately aloft of the boundary layer were generally not as strong on tornado days in tornado seasons following El Niño years. He has hypothesized that this contributes to more widespread, rather than isolated, thunderstorm development, in which tornadoes frequently form. It is necessary to determine to what extent surface wetness in the southeast might affect low-level thermal stability. Another process that is critical to the development of very strong low-level stability is transport of high potential temperature air from above the Mexican Plateau over the central U.S. [Carlson et al., 1983]. Thus it is necessary to consider how surface conditions that may be transported over the southeast U.S. from remote locations may be affected by El Niño. It is also possible that this negative dependence is related to increased tornado frequency during La Niña, which is characterized by negative SST anomaly in the equatorial Pacific. It is well known that U.S. landfalling hurricanes of tropical origin occur more frequently during La Niña than during El Niño [Elsner et al., 1999]. As tornadoes are often observed during hurricane landfalls, this correlation would suggest an increased risk for tornadoes in the southeastern U.S. during the fall months under La Niña conditions.

Finally, we mention that although the model used in this analysis is not definitive, the statistical approach is flexible. One of our goals with this study was to demonstrate that the hierarchical Bayesian approach to modeling complicated spatiotemporal processes can provide useful insight in climatological and meteorological diagnostic analyses. More importantly, this methodology can easily be adapted to more complicated scenarios. For example, a continuing area of research involves using such a hierarchical model to explicitly account for observational uncertainty.
and sampling bias in the tornado counts as well as in the residual spatiotemporal correlation. Sampling bias is made evident in this model by the correspondence between high values of the spatial mean and \( \beta_3 \) with urban areas and regions of relatively high population density. Results from such studies will be presented elsewhere. The important point is that this approach allows one to account for uncertainty in data, model, process, and parameters when performing inference on complicated physical processes, such as tornado counts.

**Appendix A: MCMC (Gibbs Sampler)**

(55) Consider the joint distribution of several variables denoted by \( w_1, \ldots, w_k \); namely, \( [w_1, \ldots, w_k] \). For example, let this distribution represent the posterior distribution (20). Next, as is true in our case, assume that \( [w_1, \ldots, w_k] \) is too complex to (1) implement mathematical formulas to find the normalizing constant or to find marginal distributions of selected variables and (2) to simulate from directly. MCMC sampling approaches can be used to obtain samples from such distributions indirectly. Specifically, rather than compute the posterior directly, one computes successive simulations from a Markov chain constructed in a fashion that permits the assertion that its stationary distribution coincides with the target posterior. After some burn-in time, realizations of the chain are viewed as simulated samples from the posterior distribution. Note that these samples are not independent. Generated samples are then used to estimate features of interest. One algorithm for producing a Markov chain that has the correct properties is the Gibbs sampler (see, e.g., Robert and Casella [1999] for an overview). This algorithm is described as follows. (1) Derive the “full-conditional distributions” for each parameter:

\[
[w_1 | w_2, \ldots, w_k], [w_2 | w_1, w_3, \ldots, w_{k-1}], \ldots, [w_k | w_1, \ldots, w_{k-1}];
\]

the structure of the hierarchical model typically makes the formulation of these full-conditionals comparatively easy. (2) Select starting values: \( (w_1^{(0)}, \ldots, w_k^{(0)}) \). (3) Iteratively sample from the full-conditional distributions as follows: Given the current state of the chain \( (w_1^{(j)}, \ldots, w_k^{(j)}) \), generate the next state according to

\[
\begin{align*}
&w_1^{(j+1)} \sim [w_1 | w_2^{(j)}, \ldots, w_k^{(j)}]; \\
&w_2^{(j+1)} \sim [w_2 | w_1^{(j+1)}, w_3^{(j)}, \ldots, w_k^{(j)}]; \\
&\vdots \\
&w_k^{(j+1)} \sim [w_k | w_1^{(j+1)}, \ldots, w_{k-1}^{(j+1)}].
\end{align*}
\]

(4) Discarding the first \( b \) iterates (so-called “burn-in” period), the following set of \( m \) Gibbs iterations gives

\[
[w_1^{(b+1)}, \ldots, w_k^{(b+1)}], [w_1^{(b+2)}, \ldots, w_k^{(b+2)}], \ldots, [w_1^{(b+m)}, \ldots, w_k^{(b+m)}].
\]

(5) Monte Carlo estimation approaches [e.g., Ripley, 1987] are applied to this sample to obtain estimates.

(55) While theory implies that the Markov chain is guaranteed to converge to the appropriate stationary distribution, implementation issues arise in practice. For example, decisions have to be made to choose starting values and the values of \( b \) and \( m \), and one has to try to ascertain when the Markov chain has indeed converged to the stationary distribution. See Gilks et al. [1996] and Robert and Casella [1999] for discussion of these issues.

**References**


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